EPFL– Spring 2025
**SOLUTIONS: Series 6**

# Differential Geometry III: Riemannian geometry

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**6.1** (*Naturality of the exponential map*). Let $(M, g_{\mathcal{M}})$ and $(\mathcal{N}, g_{\mathcal{N}})$ be smooth Riemannian manifolds and let $\Phi : \mathcal{M} \to \mathcal{N}$ be an isometry. For any $p \in \mathcal{M}$, prove that the following diagram is commutative:

$$\begin{array}{ccc} T_p\mathcal{M} & \xrightarrow{d_p\Phi} & T_{\Phi(p)}\mathcal{N} \\ \big\downarrow{\scriptstyle \exp_p} & & \big\downarrow{\scriptstyle \exp_{\Phi(p)}} \\ \mathcal{M} & \xrightarrow{\quad\Phi\quad} & \mathcal{N} \end{array} \tag{1}$$

**Solution.** We will first show that, if $\nabla^{(\mathcal{M})}$ and $\nabla^{(\mathcal{N})}$ are, respectively, the Levi-Civita connections of the metrics $g_{\mathcal{M}}$ and $g_{\mathcal{N}}$, then $\Phi$ "commutes" with covariant differentiation, i.e. for any $X, Y \in \Gamma(\mathcal{M})$, we have

$$\Phi_*\big(\nabla^{(\mathcal{M})}_X Y\big) = \nabla^{(\mathcal{N})}_{\Phi_* X}(\Phi_* Y),$$

where $\Phi_*(W) = d\Phi(W)$ denotes the push-forward via the differential of $\Phi$. The above is equivalent to the statement that, for any $X, Y, Z \in \Gamma(\mathcal{M})$: $X, Y \in \Gamma(\mathcal{M})$ and $W \in \Gamma(\mathcal{N})$:

$$g_{\mathcal{N}}\Big(\Phi_*\big(\nabla^{(\mathcal{M})}_X Y\big), W\Big) = g_{\mathcal{N}}\Big(\nabla^{(\mathcal{N})}_{\Phi_* X}(\Phi_* Y), W\Big),$$

which, since $\Phi$ is an isometry, can be reexpressed as follows (for $W = \Phi_* Z$ for any $Z \in \Gamma(\mathcal{M})$):

$$g_{\mathcal{M}}\Big(\nabla^{(\mathcal{M})}_X Y, Z\Big) = g_{\mathcal{N}}\Big(\nabla^{(\mathcal{N})}_{\Phi_* X}(\Phi_* Y), \Phi_* Z\Big). \tag{2}$$

In order to show (2), we will use the formula of Koszul

$$\begin{aligned} 2g\big(\nabla_U V, W\big) = U\big(g(V,W)\big) &+ V\big(g(U,W)\big) - W\big(g(U,V)\big) \\ &- g\big([V,W],U\big) - g\big([U,W],V\big) + g\big([U,V],W\big) \end{aligned} \tag{3}$$

expressing the Levi-Civita connection $\nabla$ in terms of the corresponding metric $g$ and use the fact that an isometry between two Riemannian manifolds should preserve the corresponding expressions for $\nabla$. More precisely, using (3) for $(g, U, V, W) = (g_{\mathcal{M}}, X, Y, Z)$ and $(g, U, V, W) = (g_{\mathcal{N}}, \Phi_* X, \Phi_* Y, \Phi_* Z)$ and noting that the corresponding right hand sides are equal since $\Phi$ is an isometry,[1] we infer that the left hand sides should also be the same (and thus (2) holds).

For any $p \in \mathcal{M}$ and $v \in T_p\mathcal{M}$, let $\gamma_{p,v}$ be the maximal geodesic of $g_{\mathcal{M}}$ satisfying $\gamma_{p,v}(0) = p$ and $\dot{\gamma}_{p,v}(0) = v$ (recall that the $\exp_p$ map satisfies $\exp_p(v) = \gamma_{p,v}(1)$). We will use the notation $\tilde{\gamma}$ for the respective geodesics on $(\mathcal{N}, g_{\mathcal{N}})$. The commutativity of the diagram (1) is then equivalent to the statement that, for any $p \in \mathcal{M}$ and $v \in T_p\mathcal{M}$:

$$\Phi\big(\gamma_{p,v}(1)\big) = \tilde{\gamma}_{\Phi(p), \Phi_*(v)}(1).$$

The above statement will follow if we show that the curve

$$\bar{\gamma} \doteq \Phi \circ \gamma_{p,v}$$

[1] Here, we also use the fact that, more generally, for any smooth map $\Phi : \mathcal{M} \to \mathcal{N}$, $\Phi_*([X,Y]) = [\Phi_* X, \Phi_* Y]$ and $(\Phi_* X)(f) = X(f \circ \Phi)$.

is a geodesic of $(\mathcal{N}, g_{\mathcal{N}})$ satisfying

$$\bar{\gamma}(0) = \Phi(p) \quad \text{and} \quad \dot{\bar{\gamma}}(0) = \Phi_* v \tag{4}$$

(since, in this case, the uniqueness theorem for the initial value problem for the geodesic equation would imply that $\bar{\gamma} = \tilde{\gamma}_{\Phi(p),\Phi_*(v)}$). The relations (4) follow easily from the definition of $\bar{\gamma}$; thus, it remains to show that $\bar{\gamma}$ is a geodesic, i.e.

$$\nabla^{(\mathcal{N})}_{\dot{\bar{\gamma}}} \dot{\bar{\gamma}} = 0.$$

We can readily compute using (2):

$$\nabla^{(\mathcal{N})}_{\dot{\bar{\gamma}}} \dot{\bar{\gamma}} = \nabla^{(\mathcal{N})}_{\Phi_* \dot{\gamma}_{p,v}} (\Phi_* \dot{\gamma}_{p,v}) = \Phi_* \Big( \nabla^{(\mathcal{M})}_{\dot{\gamma}_{p,v}} \dot{\gamma}_{p,v} \Big) = 0$$

(since $\gamma_{p,v}$ is a geodesic).

**6.2** Let $(\mathcal{M}, g)$ be a smooth *connected* Riemannian manifold.

(a) Suppose that $\Phi_1, \Phi_2 : \mathcal{M} \to \mathcal{M}$ are two isometries such that, for some $p \in \mathcal{M}$:

$$\Phi_1(p) = \Phi_2(p) \quad \text{and} \quad d_p\Phi_1 = d_p\Phi_2.$$

Prove that $\Phi_1 = \Phi_2$.

(b) Let $X \in \Gamma(\mathcal{M})$ be a Killing vector field of $(\mathcal{M}, g)$ for which there exists a point $p \in \mathcal{M}$ such that

$$X|_p = 0, \quad \nabla X|_p = 0.$$

Prove that $X = 0$.

**Solution.** (a) Let $\mathcal{K} \subset \mathcal{M}$ be the set of points $q \in \mathcal{M}$ such that $\Phi_1(q) = \Phi_2(q)$ and $d_q\Phi_1 = d_q\Phi_2$. We want to show that $\mathcal{K} = \mathcal{M}$. To this end, since $\mathcal{M}$ is connected, it suffices to show that $\mathcal{K}$ is a non-empty, open and closed subset of $\mathcal{M}$. Since $p \in \mathcal{K}$, we already know that $\mathcal{K} \neq \emptyset$; moreover, since $\Phi_1$, $\Phi_2$ are smooth maps, $\mathcal{K}$ is a closed set. Therefore, it only remains to show that $\mathcal{K}$ is an open subset of $\mathcal{M}$. Without loss of generality, we will assume that

$$\Phi_2 = \text{Id}$$

(since, otherwise, we can compare the maps $\Phi_1 \circ \Phi_2^{-1}$ and Id in place of $\Phi_1$ and $\Phi_2$).

Let $q \in \mathcal{K}$ and $v \in T_q\mathcal{M}$ be such that $v$ belongs to the domain of definition of $\exp_q$. Using Ex. 6.1, the assumption that $\Phi_1$ is an isometry of $(\mathcal{M}, g)$ implies that

$$\Phi_1\big(\exp_q(v)\big) = \exp_{\Phi_1(q)}\big(d_q\Phi_1(v)\big).$$

Our assumption that $q \in \mathcal{K}$ (and $\Phi_2 = \text{Id}$) implies that $\Phi_1(q) = q$ and $d_q\Phi_1(v) = v$; therefore,

$$\Phi_1\big(\exp_q(v)\big) = \exp_q(v).$$

Therefore, we infer that

$$\Phi_1(z) = z \quad \text{for all } z \text{ in the image of } \exp_q : \Omega_q \subset T_q\mathcal{M} \to \mathcal{M}.$$

In view of the fact that $\exp_q$ is a local diffeomorphism around $0 \in T_q\mathcal{M}$, we deduce that

$$\Phi_1(z) = z \quad \text{for all } z \text{ in an open neighborhood } \mathcal{U} \text{ of } q.$$

As a consequence, $d_z\Phi_1 = \mathrm{Id}$ for all $z \in \mathcal{U}$ and, therefore, $\mathcal{U} \subset \mathcal{K}$. Since $\mathcal{K}$ contains an open neighborhood around each of its points, we infer that $\mathcal{K}$ is an open subset of $\mathcal{M}$. Thus, $\mathcal{K} = \mathcal{M}$.

(b) Let us define similarly as before $\mathcal{K}$ to be the subset of $\mathcal{M}$ on which $X = 0$ and $\nabla X = 0$; since $\mathcal{K}$ is clearly non-empty ($p$ belongs to $K$) and closed (since $X$ is a smooth vector field), it suffices to show that $\mathcal{K}$ is open.

Let $q$ be a point in $\mathcal{K}$ and let $\mathcal{U}$ be an open neighborhood of $q$ in $\mathcal{M}$ and $\delta > 0$ such that the flow map $\Phi_t : \mathcal{U} \to \mathcal{M}$ of $X$ is defined for $t \in (-\delta, \delta)$. Recall that that, for any $z \in \mathcal{U}$, the integral curve $t \to \Phi_t(z)$ of $X$ is the unique solution of the initial value problem

$$\begin{cases} \frac{d}{dt}\big(\Phi_t(z)\big) = X|_{\Phi_t(z)}, \\ \Phi_0(z) = z. \end{cases} \tag{5}$$

Moreover, $\Phi_t$ is a semigroup in the following sense: For any $t_1, t_2 \in (-\delta, \delta)$ such that $t_1 + t_2 \in (-\delta, \delta)$ and any $z \in \mathcal{U}$, we have

$$\Phi_{t_1+t_2}(z) = \Phi_{t_1}(\Phi_{t_2}(z)) = \Phi_{t_2}(\Phi_{t_1}(z)). \tag{6}$$

Using the formula for the derivative of the composition of two functions, we can compute that the differential of $\Phi_t$ satisfies for any $z \in \mathcal{U}$ and $v \in T_z\mathcal{M}$:

$$d_z\Phi_{t_1+t_2}(v) = d_{\Phi_{t_2}(z)}\Phi_{t_1}\big(d_z\Phi_{t_2}(v)\big). \tag{7}$$

Note also that our assumption that $X$ is Killing is equivalent to the statement that $\Phi_t : \mathcal{U} \to \Phi_t(\mathcal{U}) \subset \mathcal{M}$ is an isometry for all $t \in (-\delta, \delta)$.

Since $X|_q = 0$, we deduce that

$$\Phi_t(q) = q \quad \text{for all } t \in (-\delta, \delta)$$

(it is easy to check that the constant curve $t \to q$ satisfies (5)). Therefore, the pushforward map $d_q\Phi_t = (\Phi_t)_*|_q$ maps $T_q\mathcal{M}$ to $T_q\mathcal{M}$. Using the definition of the Lie derivative of $X$, we can readily calculate that, for any $Y \in \Gamma(\mathcal{M})$:

$$\begin{aligned} \mathcal{L}_X Y|_q &= \lim_{\tau \to 0}\Big(\frac{1}{\tau}\big(d_{\Phi_\tau(q)}\Phi_{-\tau}(Y|_{\Phi_\tau(q)}) - Y|_q\big)\Big) \\ &= \lim_{\tau \to 0}\Big(\frac{1}{\tau}\big(d_q\Phi_{-\tau}(Y|_q) - Y|_q\big)\Big) \\ &= \Big(\frac{d}{dt}\big(d_q\Phi_{-\tau}\big)|_{t=0}\Big)Y|_q. \end{aligned}$$

Recall that $\mathcal{L}_X Y = [X,Y] = \nabla_X Y - \nabla_Y X$ (the last equality following from the fact that the Levi-Civita connection of $g$ is torsion-free). Since $q \in \mathcal{K}$ and, therefore, $X|_q = 0$ and $\nabla_Y X|_q = 0$, we infer that $\mathcal{L}_X Y|_q = 0$ and, thus,

$$\Big(\frac{d}{dt}\big(d_q\Phi_t\big)\Big|_{t=0}\Big)v = 0 \quad \text{for all } v \in T_q\mathcal{M}. \tag{8}$$

Using the identity (7) for $z = q$, we obtain (since $\Phi_{t_2}(q) = q$):

$$d_q\Phi_{t_1+t_2}(v) = d_q\Phi_{t_1}\big(d_q\Phi_{t_2}(v)\big).$$

Differentiating the above relation with respect to $t_1$ and then setting $t_1 = 0$ and $t_2 = t$, we obtain for any $t \in (-\delta,\delta)$ and $v \in T_q\mathcal{M}$:

$$\frac{d}{d\tau}\Big(d_q\Phi_\tau\Big)|_{\tau=t}(v) = \frac{d}{d\tau}\Big(d_q\Phi_\tau\Big)|_{\tau=0}\big(d_q\Phi_t(v)\big).$$

Therefore, using (8) for the right hand side, we infer that, for any $t \in (-\delta,\delta)$:

$$\Big(\frac{d}{d\tau}\big(d_q\Phi_\tau\big)\Big|_{\tau=t}\Big)v = 0 \quad \text{for all } v \in T_q\mathcal{M}.$$

Therefore, integrating the above equation in $t$ and using the fact that $d_q\Phi_0 = \mathrm{Id}$, we obtain

$$d_q\Phi_t = \mathrm{Id} \quad \text{for all } t \in (-\delta,\delta)$$

Therefore, in view of the fact that $\Phi_t : \mathcal{U} \to \Phi_t(\mathcal{U}) \subset \mathcal{M}$ is an isometry, arguing as in the proof of part (a) (namely noticing that the image of the exponential map $\exp_q$ is fixed under the action of $\Phi_t$) we infer that $\Phi_t(z) = z$ for $z$ in an open neighborhood $\mathcal{V}$ of $q$ for all $t \in (-\delta,\delta)$. Therefore, the vector field $X$ also vanishes on $\mathcal{V}$, proving that $\mathcal{V} \subset \mathcal{K}$. Therefore, $\mathcal{K}$ is open.

**6.3** Let $(\mathcal{M}, g)$ be a connected Riemannian manifold, and let $N \subset \mathcal{M}$ be a smooth submanifold of $\mathcal{M}$.

(a) For any $p \in \mathcal{M}$, we will define the distance of $p$ from $\mathcal{N}$ to be

$$d(p,\mathcal{N}) = \inf\big\{\ell(\gamma) : \gamma : [0,1] \to \mathcal{M} \text{ is a } C^1 \text{ curve}, \gamma(0) = p, \gamma(1) \in \mathcal{N}\big\}.$$

Assume that, for a given $p \in \mathcal{M}$, a minimizer for $d(p,\mathcal{N})$ exists, i.e. there exists a $C^1$ curve $\gamma : [0,1] \to \mathcal{M}$ such that $\gamma(0) = p$, $\gamma(1) = q \in \mathcal{N}$ and

$$\ell(\gamma) = d(p,\mathcal{N}).$$

Show that $\gamma$ is a geodesic of $(\mathcal{M}, g)$ and $\dot\gamma(1)$ is normal to $T_q\mathcal{N}$.

(b) Let $q_1, q_2$ be two points on $\mathcal{N}$ and let $\gamma : [0,1] \to \mathcal{N}$ be a $C^1$ curve such that $\gamma(0) = q_1$, $\gamma(1) = q_2$ and $\ell(\gamma)$ is minimal among all curves connecting $q_1$ to $q_2$ in $\mathcal{N}$, i.e.

$$\ell(\gamma) = \min\big\{\ell(\bar{\gamma}) : \bar{\gamma} : [0,1] \to \mathcal{N},\, \bar{\gamma}(0) = q_1,\, \bar{\gamma}(1) = q_2\big\}$$

Prove that, for any $t \in [0,1]$, there exists a parametrization of $\gamma$ for which

$$\nabla_{\dot{\gamma}(t)}\dot{\gamma}(t) \text{ is orthogonal to } T_{\gamma(t)}\mathcal{N} \subset T_{\gamma(t)}\mathcal{M}$$

(where $\nabla$ is the Levi-Civita connection of $(\mathcal{M}, g)$).

**Solution.** (a) By reparametrizing the curve $\gamma$, we can assume without loss of generality that $\|\dot{\gamma}\|$ is constant in $t$. Let $\phi_s : [0,1] \to \mathcal{M}$, $s \in (-\delta, \delta)$ be a smooth variation of $\gamma$ (i.e. $\phi_0(t) = \gamma(t)$) such that $\phi_s(0) = \gamma(0) = p$ and $\phi_s(1) \in \mathcal{N}$ for all $s \in (-\delta, \delta)$. Let also $X = \frac{\partial \phi_s}{\partial s}\Big|_{s=0}$ be the variation vector field along $\gamma$. Note that our assumptions on $\phi_s$ imply that

$$X|_{t=0} = 0 \quad \text{and} \quad X|_{t=1} \in T_{\gamma(1)}\mathcal{N}.$$

Moreover, our assumption that $\gamma$ minimizes the length among all curves connecting $q$ to $\mathcal{N}$ implies that

$$\frac{d}{ds}\ell(\phi_s)\big|_{s=0} = 0.$$

Using the formula for the variation of the length, we obtain:

$$\frac{1}{\ell(\gamma)}\Big(g\big(X, \dot{\gamma}(t)\big)\big|_{t=0}^{1} - \int_0^1 g\big(X, \nabla_{\dot{\gamma}}\dot{\gamma}\big)\, dt\Big) = 0$$

and, therefore (since $X|_{t=0} = 0$):

$$g\big(X|_{t=1}, \dot{\gamma}(1)\big) - \int_0^1 g\big(X, \nabla_{\dot{\gamma}}\dot{\gamma}\big)\, dt = 0. \tag{9}$$

In order to show that $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$ and $\dot{\gamma}(1) \perp T_{\gamma(1)}\mathcal{N}$, we will use two different kinds of variations:

1. For any smooth variation $\phi_s$ satisfying $\phi_s(1) = \gamma(1)$ for all $s \in (-\delta, \delta)$ (and, therefore, $X|_{t=1} = 0$), the relation (9) gives:

$$\int_0^1 g\big(X, \nabla_{\dot{\gamma}}\dot{\gamma}\big)\, dt = 0.$$

As we mentioned in class *any* smooth vector field $X$ along $\gamma$ with $X|_{t=0} = 0$ and $X|_{t=1} = 0$ can be written as the variation vector field at $s = 0$ of a smooth variation $\phi_s$ of $\gamma$ fixing $\gamma(0)$ and $\gamma(1)$ (this can be easily checked in local coordinates). Therefore, if $\chi : [0,1] \to [0, +\infty)$ is a smooth function with $\chi(0) = \chi(1) = 0$ and $\chi(t) > 0$ for $t \in (0,1)$, choosing $X(t) = \chi(t)\nabla_{\dot{\gamma}(t)}\dot{\gamma}(t)$ we obtain

$$\int_0^1 \chi(t)\|\nabla_{\dot{\gamma}(t)}\dot{\gamma}(t)\|^2\, dt = 0,$$

i.e. (since $\dot{\gamma}$ is smooth) that $\gamma$ is a geodesic:

$$\nabla_{\dot{\gamma}}\dot{\gamma} = 0. \tag{10}$$

2. Let $\xi \in T_{\gamma(1)}\mathcal{N}$. We can extend (in a non-unique way) $\xi$ to a vector field $X$ along $\gamma$ with $X|_{t=0} = 0$ and such that $X$ is supported only inside a neighborhood $\mathcal{U}$ of $\gamma(1)$ covered by a coordinate chart. It is easy to see (by transfering this problem on $\mathbb{R}^n$ via the local cooridnates on $\mathcal{U}$) that such an $X$ can be expressed as the variation vector field of a smooth variation $\phi_s$ of $\gamma$ satisfying $\phi_s(0) = \gamma(0)$ and $\phi_s(1) \in \mathcal{N}$ (with $\frac{\partial\phi_s(1)}{\partial_s}|_{s=0} = \xi$). Therefore, applying the relation (9) and using (10) for the second term, we obtain

$$g\big(\xi, \dot{\gamma}(1)\big) - 0 = 0.$$

Since this is true for any $\xi \in T_{\gamma(1)}\mathcal{N}$, we infer that $\dot{\gamma}(1) \perp T_{\gamma(1)}\mathcal{N}$.

(b) Let us reparametrize $\gamma$, as before, so that $\|\dot{\gamma}(t)\|$ is constant in $t \in [0,1]$. Our aim is to show that, with this parametrization, we have for any $t_0 \in (0,1)$ and any $\xi \in T_{\gamma(t_0)}\mathcal{N}$:

$$g\big(\xi, \nabla_{\dot{\gamma}(t_0)}\dot{\gamma}(t_0)\big) = 0. \tag{11}$$

Let $\phi_s : [0,1] \to \mathcal{N}$, $s \in (-\delta, \delta)$ by a smooth variation of $\gamma$ through curves that lie inside $\mathcal{N}$, satisfying in addition

$$\phi_s(0) = \gamma(0) \quad \text{and} \quad \phi_s(1) = \gamma(1).$$

Our assumption that $\gamma$ minimizes the length among such curves implies that

$$\frac{d}{ds}\ell(\phi_s)\big|_{s=0} = 0.$$

If $X = \frac{\partial\phi_s}{\partial s}\big|_{s=0}$ is the associated variation vector field along $\gamma$, the formula for the variation of the length of $\gamma$ becomes in this case:

$$\int_0^1 g\big(X, \nabla_{\dot{\gamma}}\dot{\gamma}\big)\, dt = 0. \tag{12}$$

As before, we can easily see (by working, for instance, in local coordinates around each point in $\gamma$) that, for any smooth vector field $X$ along $\gamma$ that satisfies

$$X|_{\gamma(0)} = 0, \quad X|_{\gamma(1)} = 0 \quad \text{and} \quad X|_{\gamma(t)} \in T_{\gamma(t)}\mathcal{N}, \tag{13}$$

there exists a (non-unique) smooth variation $\phi_s : [0,1] \to \mathcal{N}$, $s \in (-\delta, \delta)$ of $\gamma$ satisfying $\phi_s(0) = \gamma(0)$ and $\phi_s(1) = \gamma(1)$. Therefore, (15) is true for any vector field $X$ along $\gamma$ satisfying (13).

We will now proceed to show that (15) implies that $\nabla_{\dot{\gamma}}\dot{\gamma}$ is orthogonal to $\mathcal{N}$. Assume, for the sake of contradiction, that there exists a $t_0 \in (0,1)$ and a $\xi \in T_{\gamma(t_0)}\mathcal{N}$ such that

$$g(\xi, \nabla_{\dot{\gamma}}\dot{\gamma}|_{t=t_0}) \neq 0.$$

Without loss of generality, we can assume that

$$g(\xi, \nabla_{\dot{\gamma}}\dot{\gamma}|_{t=t_0}) > 0. \tag{14}$$

Let $Y$ be a smooth vector field along $\gamma$ which is an extension of $\xi$ (i.e. $Y|_{\gamma(t_0)} = \xi$) and which is tangent to $\mathcal{N}$ (i.e. $Y|_{\gamma(t)} \in T_{\gamma(t)}\mathcal{N}$ for all $t \in [0,1]$). Let $\psi : [0,1] \to [0,+\infty)$ be a smooth cut-off function satisfying $\psi(t_0) = 1$ and such that the support of $\psi$ is inside a small enough neighborhood of $t_0$ so that

$$g(Y|_{\gamma(t)}, \nabla_{\dot\gamma}\dot\gamma(t)) > 0 \quad \text{for all } t \in \text{supp}\psi$$

(this is possible in view of (14) and the fact that $Y$ and $\nabla_{\dot\gamma}\dot\gamma$ are continuous vector fields along $\gamma$). Let us define the vector field $X$ along $\gamma$ by

$$X|_{\gamma(t)} \doteq \psi(t)Y|_{\gamma(t)}.$$

Notice that $X$ satisfies (13). Moreover, since $\psi \geqslant 0$ and $\psi(t_0) = 1 > 0$, the above condition implies that

$$\int_0^1 g\big(X, \nabla_{\dot\gamma}\dot\gamma\big)\, dt > 0 \tag{15}$$

which is a contradiction in view of (15). Therefore,

$$\nabla_{\dot\gamma}\dot\gamma(t) \perp T_{\gamma(t)}\mathcal{N} \quad \text{for all } t \in (0,1)$$

which, in view of the continuity of $\nabla_{\dot\gamma}\dot\gamma$, implies that $\nabla_{\dot\gamma}\dot\gamma(t)$ is orthogonal to $T_{\gamma(t)}\mathcal{N}$ for all $t \in [0,1]$.

**6.4** (a) Let $(\mathbb{H}^2, g_{\mathbb{H}})$ be the Poincare half plane (see also Exercise 5.2): $\mathbb{H}^2 = \big\{(x,y) \in \mathbb{R}^2 : y > 0\big\}$ and

$$g_{\mathbb{H}} = \frac{dx^2 + dy^2}{y^2}.$$

Let also $\mathbb{D}^2$ be the unit disc in $\mathbb{R}^2$, equipped with the metric

$$g_{\mathbb{D}} = 4\frac{dx^2 + dy^2}{\big(1 - x^2 - y^2\big)^2}.$$

Identifying $\mathbb{R}^2$ with $\mathbb{C}$, show that the map $\Phi : \mathbb{D}^2 \to \mathbb{H}^2$ given by

$$\Phi(z) = -i\frac{z+1}{z-1}$$

is an isometry ($(\mathbb{D}^2, g_{\mathbb{D}})$ is known as the *Poincaré disc*; both $(\mathbb{H}^2, g_{\mathbb{H}})$ and $(\mathbb{D}^2, g_{\mathbb{D}})$ are models for the hyperbolic plane).

(b) Let $p$ be a point in the hyperbolic plane. Compute the metric in polar coordinates around $p$. *(Hint: Working in the Poincaré disc model, it suffices to only consider the case when $p$ is at the origin, since any point $p \in \mathbb{D}^2$ can be mapped to any other point in $\mathbb{D}^2$ via an isometry. What are the geodesics in $(\mathbb{D}^2, g_{\mathbb{D}})$ emanating from the origin?)*

(c) How is the round metric $(\mathbb{S}^2, g_{\mathbb{S}^2})$ expressed in polar coordinates around a point $p \in \mathbb{S}^2$?

**Solution.** (a) It is easy to check that the map $\Phi : \mathbb{D}^2 \to \mathbb{H}^2$, $(x,y) \to (\bar x, \bar y) = \Big(\frac{2y}{(x-1)^2+y^2}, \frac{1-x^2-y^2}{(x-1)^2+y^2}\Big)$ is 1-1, onto and bi-continuous. Moreover, we can calculate

$$\Phi_* g_{\mathbb{H}} = \Phi_*\Big(\frac{d\bar x^2 + d\bar y^2}{\bar y^2}\Big)$$

$$
\begin{aligned}
&= \frac{\left[d\left(\frac{2y}{(x-1)^2+y^2}\right)\right]^2 + \left[d\left(\frac{1-x^2-y^2}{(x-1)^2+y^2}\right)\right]^2}{\left(\frac{1-x^2-y^2}{(x-1)^2+y^2}\right)^2} \\
&= \frac{1}{\left(\frac{1-x^2-y^2}{(x-1)^2+y^2}\right)^2}\Bigg(\Big[\frac{2}{(x-1)^2+y^2}dy - \frac{4y}{((x-1)^2+y^2)^2}\big((x-1)dx+ydy\big)\Big]^2 \\
&\qquad\qquad + \Big[-\frac{2}{(x-1)^2+y^2}\big(xdx+ydy\big) - \frac{2(1-x^2-y^2)}{((x-1)^2+y^2)^2}\big((x-1)dx+ydy\big)\Big]^2\Bigg) \\
&= \frac{1}{\left(\frac{1-x^2-y^2}{(x-1)^2+y^2}\right)^2}\left(\frac{4(dx^2+dy^2)}{((x-1)^2+y^2)^2}\right) \\
&= g_{\mathbb{D}}.
\end{aligned}
$$

Therefore, $\Phi$ is an isometry.

(b) As we saw in Exercise 5.2, the set of isometries of $(\mathbb{H}^2, g_{\mathbb{H}})$ contains all maps of the form $z \to \frac{az+b}{cz+d}$, $ad-bc>0$; therefore, for any $p_1, p_2 \in \mathbb{H}^2$, there exists an isometry $F : (\mathbb{H}^2, g_{\mathbb{H}}) \to (\mathbb{H}^2, g_{\mathbb{H}})$ such that $F(p_1) = p_2$ (i.e. $(\mathbb{H}^2, g_{\mathbb{H}})$ is *homogeneous*). As a result, the metric $g_{\mathbb{H}}$ expressed in polar coordinates around a point $p \in \mathbb{H}^2$ will have the same form independently of the chosen point $p$. For this reason, we can choose to work with the point corresponding to the origin in $(\mathbb{D}^2, g_{\mathbb{D}})$.

Let us use the notation $(x, y)$ and $(\bar{r}, \bar{\theta})$ for the standard Cartesian and radial coordinates, respectively, on $\mathbb{R}^2$ (so that $\bar{r}^2 = x^2 + y^2$ and $\tan\bar{\theta} = \frac{y}{x}$). In the $(x, y)$ coordinate system, the tangent vectors $e_1 = \frac{\partial}{\partial x}\Big|_p$ and $e_2 = \frac{\partial}{\partial y}\Big|_p$ constitute an ortnormal basis of $T_p\mathbb{D}^2$ with respect to $g_{\mathbb{D}}|_p$ (since $(g_{\mathbb{D}})_{ij}|_p = \delta_{ij}$). Therefore, we can use the coordinates on $T_p\mathbb{D}^2$ with respect to $(e_1, e_2)$ to construct a normal coordinate system in a neighborhood of $p = (0, 0)$ in $(\mathbb{D}^2, g_{\mathbb{D}})$ via the map $\exp_p$; we will use the notation $(x^1, x^2)$ for this coordinate system and $(r, \theta)$ for the associated polar coordinates (so that $r^2 = (x^1)^2 + (x^2)^2$ and $\tan\bar{\theta} = \frac{x^2}{x^1}$). Notice that, since $e_1 = \frac{\partial}{\partial x}\Big|_p$ and $e_2 = \frac{\partial}{\partial y}\Big|_p$, we have

$$
\frac{\partial}{\partial x^1}\Big|_p = \frac{\partial}{\partial x}\Big|_p \quad \text{and} \quad \frac{\partial}{\partial x^2}\Big|_p = \frac{\partial}{\partial y}\Big|_p. \tag{16}
$$

Moreover, in the $(r, \theta)$ coordinate system, the curves $\theta = \text{const}$ correspond to geodesic rays emanating from $p$. Recall that, as we saw in class, the metric $g_{\mathbb{D}}$ in polar coordinates takes the form

$$
g_{\mathbb{D}} = dr^2 + \big(b(r,\theta)\big)^2 d\theta^2,
$$

with $\lim_{r\to 0} b(r, \theta) = 0$ and $\lim_{r\to 0} \frac{b(r,\theta)}{r} = 1$. Our aim is to express $r, \theta$ as functions of the background coordinates $\bar{r}, \bar{\theta}$ on $\mathbb{D}^2 \subset \mathbb{R}^2$ and compute $b(r, \theta)$. To this end, we want to make use of the fact that $(\mathbb{D}^2, g_{\mathbb{D}})$ expressed in the $(\bar{r}, \bar{\theta})$ coordinate system is rotationally symmetric to infer that $\theta = \bar{\theta}$ and that $\bar{r}$ and $b$ are functions only of $\bar{r}$ (and not of $\theta$). Even though this statement should be intuitively clear, let us try to set up this argument in detail.

It is easy to verify that the geodesics of $(\mathbb{D}^2, g_{\mathbb{D}})$ emanating from the origin are straight line segments in $\mathbb{D}^2$. Therefore, the curves $\{\theta = \text{const}\}$ are the same as the curves $\{\bar{\theta} = \text{const}\}$, i.e. $\theta = \theta(\bar{r}, \bar{\theta})$ is a function *only* of $\bar{\theta}$. We will now show that this implies that $\theta = \bar{\theta}$: The condition (16) implies that the Jacobian matrix of the transformation matrix $(x, y) \to (\bar{x}, \bar{y})$ satisfies

$$\begin{bmatrix} \partial_x x^1 & \partial_x x^2 \\ \partial_y x^1 & \partial_y x^2 \end{bmatrix} \xrightarrow{(x,y)\to(0,0)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus, using the fact that $\theta = \text{Arctan}(\frac{x^2}{x^1})$ and $\bar{\theta} = \text{Arctan}(\frac{y}{x})$, we infer that

$$\lim_{\bar{r}\to 0} \frac{\theta}{\bar{\theta}} = 1.$$

The fact that $\theta = \theta(\bar{\theta})$ then implies that

$$\theta = \bar{\theta}.$$

We will now seek an expression for $r = r(\bar{r}, \bar{\theta})$. Recall that the point $q$ in $(\mathbb{D}^2, g_{\mathbb{D}}$ corresponding to the polar coordinate pair $(r, \theta)$ is simply

$$q = \exp_p \big( r \cos\theta e_1 + r \sin\theta e_2 \big).$$

In particular, for any $\rho > 0$, the set $\{r = \rho\}$ in $\mathbb{D}$ is the images under $\exp_p$ of the set $S^{(p)}_\rho = \{v = (v^1, v^2) \in T_p\mathbb{D}^2 : (v^1)^2 + (v^2)^2 = \rho^2\}$ (where $(v^1, v^2)$ are the coordinates of $v$ in the orthonormal basis $\{e_1, e_2\} = \{\partial_x|_p, \partial_y|_p\}$). The following observation is crucial: In the $(\bar{r}, \bar{\theta})$ coordinate system, the metric $g_{\mathbb{D}}$ takes the form

$$g_{\mathbb{D}} = \frac{4}{(1-\bar{r}^2)^2} \big( d\bar{r}^2 + \bar{r}^2 d\bar{\theta} \big), \tag{17}$$

i.e. the coefficients of the metric are *independent* of $\bar{\theta}$, hence the rotations $\Phi_\lambda : (\bar{r}, \bar{\theta}) \to (\bar{r}, \bar{\theta} + \lambda)$ are *isometries* for $g_{\mathbb{D}}$. Using the fact that isometries map geodesics to geodesics (see Ex. 6.1), and $\Phi_*|_p$ maps $S_\rho$ to $S_\rho$, we infer that, for any $\rho > 0$, the set $\{r = \rho\}$ is invariant under the rotations $\Phi_\lambda$, $\lambda \in \mathbb{R}$. Since these rotations also leave the circles $\{\bar{r} = \text{const}\}$ invariant, we infer that the curves $\{r = \text{const}\}$ and $\{\bar{r} = \text{const}\}$ are the same, i.e. $r$ is a function only of $\bar{r}$. Therefore, since $r = r(\bar{r})$ and $\theta = \bar{\theta}$, in the $(r, \theta)$ coordinate system the isometries $\Phi_\lambda$ also take the form $(r, \theta) \to (r, \theta + \lambda)$; we deduce that, in the polar $(r, \theta)$ coordinate system, the coefficients of $g_{\mathbb{D}}$ should be independent of $\theta$, i.e. that $b$ is a function only of $r$. Thus, we have the following expressions for $g_{\mathbb{D}}$ in the coordinate systems $(r, \theta)$ and $(\bar{r}, \bar{\theta}) = (\bar{r}, \theta)$:

$$g_{\mathbb{D}} = dr^2 + (b(r))^2 d\theta^2 = \big(\frac{dr}{d\bar{r}}\big)^2 d\bar{r}^2 + (b(r))^2 d\theta^2$$

and, in view of (17):

$$g_{\mathbb{D}} = \frac{4}{(1-\bar{r}^2)^2} (d\bar{r}^2 + \bar{r}^2 d\theta^2).$$

We therefore infer that

$$\frac{dr}{d\bar{r}} = \frac{2}{1-\bar{r}^2} \quad \text{and} \quad b\big(r(\bar{r})\big) = \frac{2\bar{r}}{1-\bar{r}^2}$$

from which we obtain

$$r(\bar{r}) = \log\Big(\frac{1+\bar{r}}{1-\bar{r}}\Big) \quad \text{and} \quad b(r) = \sinh(r).$$

Thus, in polar coordinates $(r, \theta)$ around $p = (0, 0)$, $g_{\mathbb{D}}$ takes the form:

$$g_{\mathbb{D}} = dr^2 + (\sinh r)^2 d\theta^2.$$

Notice that $(r, \theta) \in (0, +\infty) \times [0, 2\pi)$ covers all of $\mathbb{D}^2 \setminus 0$.

(c) As in the case of the hyperbolic plane, the round sphere $(\mathbb{S}^2, g_{\mathbb{S}^2}$ is homogenenous and, therefore, the metric expressed in polar coordinates around a point $p \in \mathbb{S}^2$ will have the same form independently of the choice of $p$; we can therefore choose $p$ to be the north pole $N$). Recall that, in stereographic coordinates from $N$ (which parametrize $\mathbb{S}^2 \setminus S$ by point on the plane $\mathbb{R}^2$, see Ex. 2.3), the round metric $g_{\mathbb{S}^2}$ takes the form

$$g_{\mathbb{S}^2} = \frac{4}{(1+x^2+y^2)^2}\big(dx^2 + dy^2\big)$$

(wih $(x, y) = (0, 0)$ corresponding to $p$ and $x^2 + y^2 \to +\infty$ corresponding to $N$). In particular, switching to radial coordinates $(\bar{r}, \bar{\theta})$ on $\mathbb{R}^2$, we have

$$g_{\mathbb{S}^2} = \frac{4}{(1+\bar{r}^2)^2}\big(d\bar{r}^2 + \bar{r}^2 d\bar{\theta}^2\big). \tag{18}$$

We immediately notice that geodesics emanating from $p$ correspond, in the above coordinate system, to straight lines $\bar{\theta} = \text{const}$ and that the metric $g_{\mathbb{S}^2}$ is invariant under rotations $(\bar{r}, \bar{\theta}) \to (\bar{r}, \bar{\theta} + \lambda)$. Therefore, arguing exactly as in the case of the hyperbolic plance, we infer that the polar coordinate system $(r, \theta)$ around $p$ satisfies $\theta = \bar{\theta}$ and $r = r(\bar{r})$ and that $b(r, \theta)$ is a function of $r$ only, i.e.

$$g_{\mathbb{S}^2} = dr^2 + (b(r))^2 d\theta^2.$$

Comparing the above expression with (18), we deduce that

$$\frac{dr}{d\bar{r}} = \frac{2}{1+\bar{r}^2} \quad \text{and} \quad b(r(\bar{r})) = \frac{2\bar{r}}{1+\bar{r}^2},$$

i.e. that

$$r(\bar{r}) = 2\arctan\bar{r} \quad \text{and} \quad b(r) = \sin(r).$$

Thus,

$$g_{\mathbb{S}^2} = dr^2 + \sin^2 r d\theta^2$$

and $(r, \theta) \in (0, \pi) \times [0, 2\pi)$ covers $\mathbb{S}^2 \setminus \{N, S\}$

**Remak.** Notice the analogy with the corresponding expression for the hyperbolic metric.